Searching for Simpler Models of Astrophysical Synchronization

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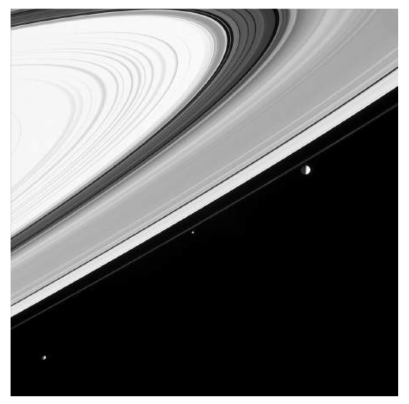
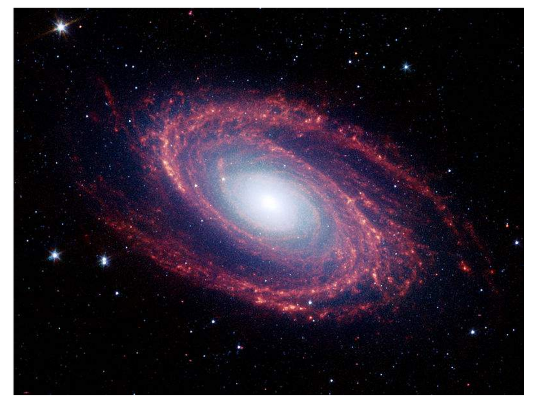
In Predicting Synchronization, Is Less More?



Courtesy NASA/JPL-Caltech

Satellite formation, rings, spiral arms and more could be described by synchronization models.



Courtesy NASA/JPL-Caltech

Synchronization in two- and three- body astrophysical systems is well understood using the tools of classical mechanics[1]. With more than three bodies, analytical solutions become nearly impossible, and numerical experiments require enormous computational resources. We explore the possibility of using a non-conservative model, with the goal of faster computation and simpler analytical solutions. We focus on many-body dissipative systems such as circumplanetary discs of dust.

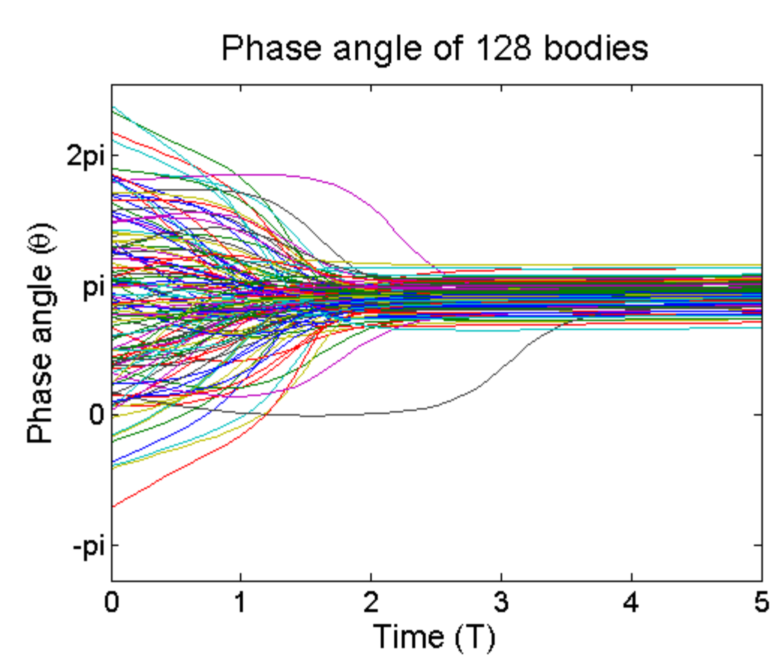
The Kuramoto Model as Motivator

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

Frequency of i^{th} object — Natural frequency of i^{th} object — Adjustments to frequency of i^{th} object due to forces from all others

The **Kuramoto model** describes the frequency of oscillations of coupled, non-linear oscillators. The natural frequency of a body or oscillator is affected by all other objects in the system[2].

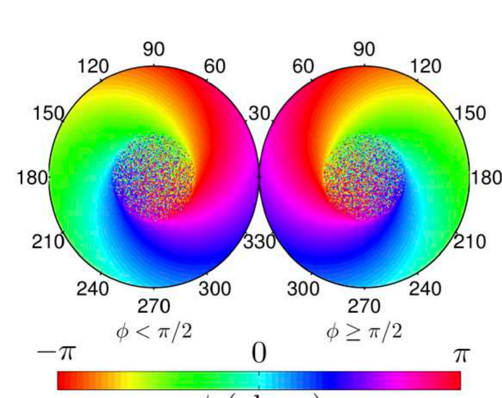
All objects in the system affect each other through different types of forces, represented by the constant K in the equation. The strength of the effects leads to clustering of various tightness, as shown in the plot at right.



Lightning bugs in York, PA, Flickr user tom.arthur, 2009

Creatures <u>and</u> objects that behave cyclically can begin out of step and then synchronize.
Example: the blinking of fireflies (*Pteroptyx cribellata, Pteroptyx malaccae*)

With modification, the Kuramoto model can give rise to **"chimera states"**, in which some fraction of the oscillators synchronize. These can appear as clusters or spirals of synchronized oscillators, which may suggest astronomical objects like galaxies.



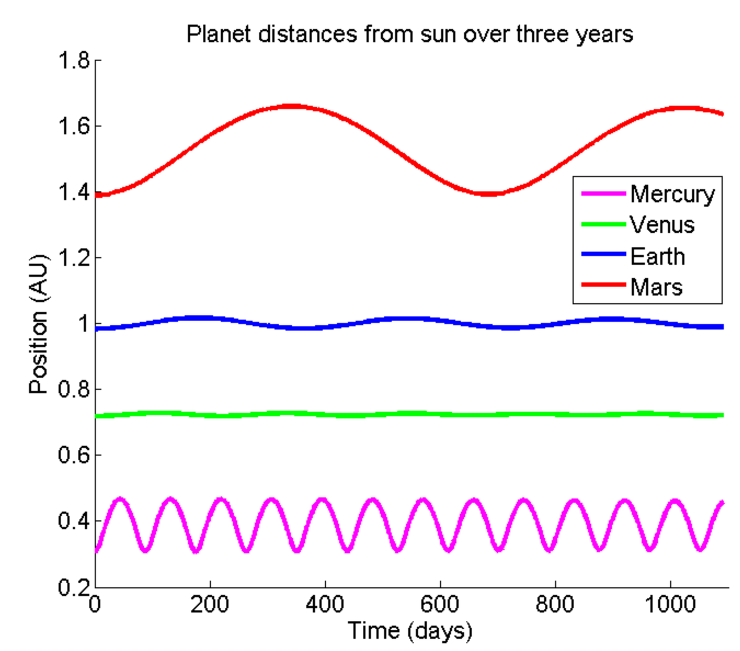
Courtesy Dr. Daniel Abrams

Customizable Simulation Code

Step 1: build N-body simulation based on traditional model.
Step 2: modify simulation to use custom mathematical model.

Initial stages of the project consisted of programming a custom *N*-body simulator in MATLAB using a 4th order Runge-Kutta method of integration to solve Newton's equation of motion (where d=|r| is the distance between two bodies):

$$\frac{d^2\vec{r}}{dt^2} = -\frac{GM}{d^3}\vec{r}$$

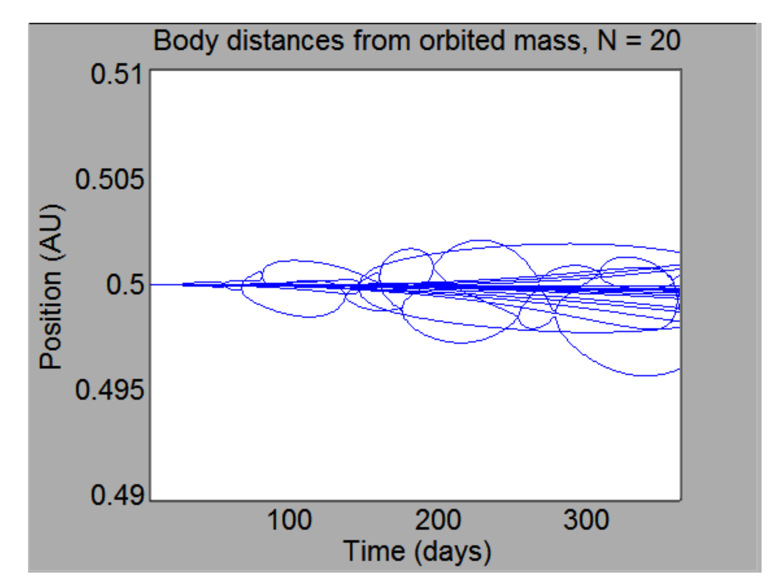


Plot showing, as proof of concept, inner planet distances from the sun over 3 years. Each planet's total distance oscillates about its mean. This plot does not show synchronization, as expected.

(Note: all 8 planets and the sun were simulated, but only four were plotted to avoid problems of scale.)

Plot showing the simulation of 20 bodies (m=10^{24} kg) at 0.5 AU from the host planet (m=10^{26} kg). Initial positions are randomly generated.

In this example, no collisions or synchronization occur due to relative sparseness of body distribution.



A New Nonlinear Model of Interaction

We are currently working to adapt the code to use a new differential equation for the changes in orbital speed of a collection of many bodies around a much larger body—for example, a dust ring around a planet. This equation will be reminiscent of the Kuramoto model:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{GK}{4R^3}\int_0^{2\pi}\frac{\rho(\theta')d\theta'}{\sin^2\left(0.5\left|\theta_i - \theta'\right|\right)}$$

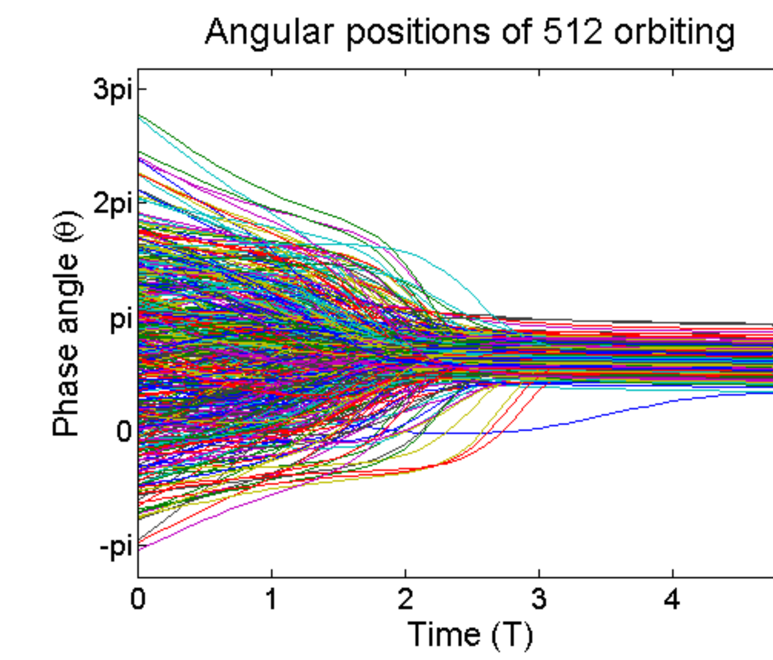
The influence of gravity as the "communicating force" is seen in the presence of G and R^3. Other features include:

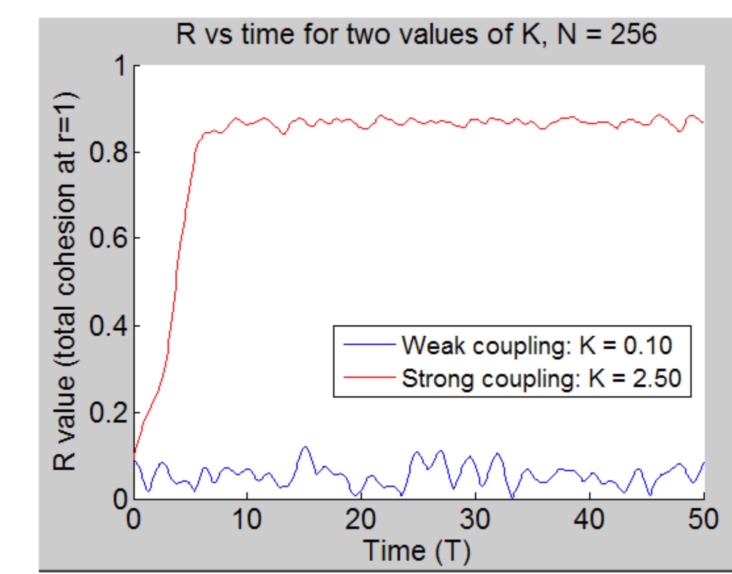
- K: a scaling constant, as in the Kuramoto model
- ρ: the angular density of bodies in the orbit
- Nonlinear status from sine function in the denominator (due to the geometry of separation of any two bodies)
- Use of an integral in lieu of a sum, consistent with large *N*

Determining Validity of the Model

The example plots from the N-body simulator show *orbital radius* over time, but to evaluate our nonlinear model, we will instead track the phase angle of each oscillator, as in the plot in the "Motivation" section.

If our model is reasonable, we expect to see something closer to the plot at right, which shows phase synchronization of all the objects. This could be the formation of a moon; multiple clusters could represent other regular structures such as spiral arms.





Synchronization can also be checked by plotting the order parameter *R*, which roughly corresponds to strength of clustering. *R*=1 is perfect synchronization. The plot at left is an example for a system obeying the Kuramoto model.

- Red line: a strongly coupled system quickly synchronizes.
- Blue line: in a weakly coupled system, objects synchronize rarely or not at all.

Future results matching the red line more closely would support our model.

Future Work

Our current goal is code optimization and implementation of our custom nonlinear model. In addition, we will continue refining the code and adding complexity, such as:

- Special treatment of collision dynamics
- Generalization to three dimensions
- Conversion to polar and spherical coordinates to track phase angles
- Expansion to large *N* simulations
  - Applications to galactic spiral arms or planetary formation
  - Ring formation around massive planets

Acknowledgments

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References

1 Gladman, Brett, D. Dane Quinn, Philip Nicholson, and Richard Rand. "Synchronous Locking of Tidally Evolving Satellites." *Icarus* 122 (1995): 166-92.
2 Strogatz, Steven H. "From Kuramoto to Crawford: Exploring the Onset of Synchronization in Populations of Coupled Oscillators." *Physica D* 143 (2000): 1-20.

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